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ABSTRACT

The Rasch model for the probability of a person's response to an item is extended to the case where this response depends on a set of scoring or category weights, in addition to person and item parameters. The maximum likelihood approach introduced by Wright for the dichotomous case is applicable here also, and it is shown to yield a unique solution. The method of estimating the three sets of parameters consists in using the person-by-category responses summed over items and the item-by-category responses summed over persons to yield a system of (N+k) m linear equations in the (N+k)m unknown parameters. (N=number of people, k=number of items, m=number of categories) They are solved iteratively using a Newton-Raphson procedure. The matrix of Nm person-by-category estimates and the matrix of km item-by-category estimates each have eigenvectors whose components are proportional to the scoring weights. The other factors yield the "ability" estimators and "easiness" estimators respectively. Computer programs for the estimation procedure have been developed. (Author)

ON AN EXTENSION OF THE RASCH MODEL TO THE CASE OF POLYCHOTOMOUSLY SCORED ITEMS.

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The introduction of the Rasch model for the probability of a person's response to a test item as a function of the person's "ability" and the item's "easiness" has opened up new vistas in mental testing. Its mathematical rigor and elegance are satisfying to the theoretician and illuminating to the test maker and evaluator. As Benjamin Wright has shown in his work (1)(2), the objectivity of testing is vastly increased by the use of such a model. Item easiness can be estimated independently of ability, and after an initial calibration of test items ability measures no longer depend on group performance, but can be obtained from high and low groups alike. This paper deals with the generalization of the model to the case of polychotomously scored items, such as attitude scales, where we deal not only with the two sets of parameters from the dichotomous case, but we have a third set of parameters- the category or scoring weights. The probability of a response in category n conditional upon the ability of the individual i and item easiness j can now be written

$$p(n \mid i,j) = \frac{e^{\theta_{in} + \epsilon_{jn}}}{\sum_{n=1}^{m} e^{\theta_{in} + \epsilon_{jn}}}$$

i=1,N (Number of persons)
j=1,k (" " items)
n=1,m (" " categories)

The categories are subject to the linear constraint that their sum is constant.

The study of this model (as in the dichotomous case) can be divided into three distinct stages: an estimation procedure for the unknown parameters, the variabilities of these and thus their possible control, and the fit of the data to the model. This paper deals only with the estimation problem, leaving the next two to a separate study.

Rasch showed (3)(4) that the mathematical properties of the dichotomous model are preserved also in this case- i.e. the sets of person or ability parameters are separable from the item or easiness parameters. One way of estimating these

parameters is suggested by the fact that certain conditional probabilities can be expressed solely in terms of one of the sets, independently of the other. This approach has been used by Andersen in his work in Denmark (5). Wright has found in the dichotomous case that a procedure based on unconditional probabilities yields the same results and is much easier to apply. Although the identity of the methods has only been shown empirically, there is reason to believe that a theoretical proof can be given. This study is based on Wright's approach.

If we think of each possible response pattern as a selection vector $X_{ijn} = (0,0,0\ldots\overset{n}{1},0,\ldots 0)$ with components x_{ijn} all equal to zero except the nth which is equal to unity, we can write the probability of a particular response pattern, i.e.

$$P(X_{ij1}, X_{ij2}, \ldots X_{ijn} \ldots X_{ijn}) = (x_{ij1}, x_{ij2} \quad x_{ijn}) = \frac{e^{\sum_n (\theta_{in} + \epsilon_{jn}) x_{ijn}}}{\sum_n e^{\theta_{in} + \epsilon_{jn}}}$$

and the joint probability for these patterns under the model is given by the likelihood function

$$L = \frac{e^{\sum_i \sum_j \sum_n (\theta_{in} + \epsilon_{jn}) x_{ijn}}}{\prod_i \prod_j \left(\sum_n e^{\theta_{in} + \epsilon_{jn}} \right)}$$

Differentiating the logarithm of this expression we show that it possesses a unique maximum.

The linear system which we obtain consists of (N+k)m equations in (N+k)m unknowns, Nm of these being person-by-category parameters and the other km item-by-category parameters. The observations or constants of the system are the marginal responses when summed over items (i.e. number of times person i chose category n) and the marginal item-by-category entries; i.e. number of times item j received response n summed over persons. The solutions of these equations involve two stages of iterations: Firstly, after some initialization, we obtain a first approximation to one of the sets of parameters, say the person-by-category ones. Then, using this approximation, we calculate the other set. Now we use this to get a second approximation to the first set, etc. The second iterative procedure

concerns the single parameters each of which is calculated by a Newton-Raphson procedure. It is shown theoretically that the Newton-Raphson method converges, and experience confirms that the convergence is rapid.

Having thus obtained an Nxm matrix of person-by-category parameters and a kxm matrix of item-by-category parameters, we find that these can each be decomposed into sums of rank 1 matrices. The corresponding terms in the two sums have one vector in common (i.e. the two matrices have common eigenvectors) and the components of these common vectors are the category weights. The remaining factors in the respective terms represent the ability estimates and the item estimates. The maximum number of terms in the two sums is m-1 (i.e. the number of eigenvectors) and represents the "dimensionality" of the test. However the terms decrease rapidly, and from the test maker's point of view it would be desirable to restrict the dimensionality to a practical minimum, e.g. two. Thus one might end up with two sets of scoring weights, one usually representing a linear ordering the other of a quadratic character. Associated with each is an ability vector and an item vector. Once items have been calibrated, it is not necessary to calculate all of these parameters, as it is shown how a person's score along the two dimensions can be obtained from the scoring weights and just his responses.

The decomposition of the matrices mentioned above is also achieved by an iterative procedure and computer programs for all the described estimation procedures have been developed.

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